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STOKES FLOW PAST SLITS AND HOLES

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Abstract—A linear shear flow is present in a fluid along one side of a fixed wall; the fluid on the other side is at rest. However, when there is a slit or a hole in the wall, then the shear flow will induce a motion in the fluid on the second side through a shearing force across the gap. We consider the Stokes flow behaviour for such a phenomenon in a number of different geometries, with particular interest in the singular solutions generated when the width of the slit, or size of the hole, is small.

1. INTRODUCTION

The purpose of the present paper is to consider how the effects of a flow along a wall can be transmitted through a slit or a hole in the wall. We assume that the fluid fills all the space on both sides of the wall, and that if it were not for the presence of the slit or hole then the fluid would be at rest on one side. When the width of the slit, or the size of the hole, is small then, whatever the nature of the flow on the external side of the wall, it can be approximated there by a linear shear flow. (The one exception would be if the external flow in the neighbourhood of the hole is a stagnation point flow; this is not considered at all in the present paper.) Further, it is sufficient to measure the effects through Stokes flow calculations—when the local Reynolds number Re is based on the length scale of the hole, we can realistically take Re to be small. Consequently, the discussion here comprises a number of relevant solutions of the equations for Stokes flow in both 2-D and 3-D.

The search for such solutions has a long and distinguished history. The classical results are well-presented in the text of Happel & Brenner (1983), and more recent results, motivated by the interest in separated Stokes flows, have been given in the survey article by O'Neill & Ranger (1979).

We are able to solve exactly the 2-D Stokes flow which measures the effect of a linear shear flow past a slit of finite width in an infinite plane wall; as the width a tends to zero a singular solution is derived, with magnitude proportional to a^2. There is no net mass flux across the slit; in fact, there is no mixing at all of the fluid on the two sides of the wall, and the total effect is transmitted by the shear stress across the slit, even when it has finite width. The flow represented by the singular solution is purely radial, being towards the slit on the upstream side of the shear flow, and away from the slit on the downstream side; there is a resultant net mass flux induced in the otherwise quiescent fluid in the same direction as the shear flow. This behaviour is complementary to the 2-D source flow through a slit in a wall.

The other basic solution obtained exactly is for the linear shear flow along an infinite plane wall which has a circular hole of finite radius c. There is no mass flux across the hole in this case either, for the same physical reasons as mentioned above. Again, the limiting case as c tends to zero can be calculated, leading to a singular solution which represents a fully radial (though asymmetric) flow in the spherical sense.

Examples of the effect of the singular behaviour in different geometries are presented for both the 2-D and 3-D situations to illustrate the possibilities present.

Finally, we present solutions for shear flow past narrow straight line slits of finite length; these represent the intermediate situations between the (2-D) slit and the (3-D) hole, and simple descriptions of the behaviours are possible to help develop a clear understanding of the attendant flows.

Although the comments so far have referred to flows in a single, uniform fluid, it is a straightforward task to extend all the results mentioned to the two-phase situation, where there are different fluids on either side of the slit or hole. Because the basic action of a slow viscous flow is just to transmit a shear stress across the slit, with the consequence that no fluid flows through the gap and there is no mixing, it can be shown that these results have an immediate applicability to the corresponding two-phase flow. In fact, the streamlines for the perturbed flow are unchanged from those found previously, only the magnitude of the velocities require adjustment, the degree depending very simply on the ratio of the viscosities. Therefore, all the results required for the two-phase flows can be written down with no further calculations.

2. FLOWS PAST INFINITE SLITS

(A) The first basic problem we consider concerns the 2-D linear shear flow past a slit in a plane. Although the calculations are reasonably straightforward, they are presented here fully because the basic ideas carry over to more detailed cases later. Specifically, the plane occupies the region $|x| > a$ of $y = 0$, where a is a non-dimensional constant; the shear flow is externally driven in the upper half-plane $y \geqslant 0$. Consequently, we must find a solution of the biharmonic equation $\nabla^4\psi = 0$ for the stream function $\psi(x, y)$ which satisfies

$$\left.\begin{aligned} \psi = \psi_y = 0 \quad \text{on} \quad y = 0, \quad |x| > a, \\ \psi \simeq y^2 \quad \text{as} \quad y \to \infty, \quad \psi \to 0 \quad \text{as} \quad y \to -\infty. \end{aligned}\right\} \qquad [1]$$

The velocities in the x- and y-directions are then given by $u = \psi_y$ and $v = -\psi_x$, respectively. The solution can actually be written down as a special case of the work of O'Neill (1977) and Wakiya (1978), who considered the slow linear shear flow over a cylindrical trough, with circular cross-section, in a plane. The trough in the lower half-space had the profile $x^2 + (y + d)^2 = a^2 + d^2 (a, d > 0)$, intersecting the plane $y = 0$ in the points $x = \pm a$ for all d; when $d \to \infty$ the particular situation posed above emerges. Their solutions used bipolar coordinates.

More directly, with the advantage of a simpler geometry, we use the fact that every function can be decomposed into even and odd components to introduce a symmetry into the problem; i.e. we write $\psi(x, y) = \psi_1(x, y) + \psi_2(x, y)$, where ψ_1 is antisymmetric and ψ_2 is symmetric in y. It is then necessary that ψ_1 and ψ_2 are biharmonic functions satisfying

$$\left.\begin{aligned} &\psi_1 = \psi_{1y} = 0 \quad \text{on} \quad y = 0, \quad |x| > a, \\ &\psi_1 = \psi_{1yy} = 0 \quad \text{on} \quad y = 0. \quad |x| < a, \\ &\psi_1 \simeq \pm\tfrac{1}{2}y^2 \quad \text{as} \quad y \to \pm\infty; \end{aligned}\right\} \qquad [2a]$$

and

$$\left.\begin{aligned} &\psi_2 = \tfrac{1}{2}\mathrm{f}\,\mathrm{sgn}\,x, \quad \psi_{2y} = 0 \quad \text{on} \quad y = 0, \quad |x| > a, \\ &\psi_{2y} = \psi_{2yyy} = 0 \quad \text{on} \quad y = 0, \quad |x| < a, \\ &\psi_2 \simeq \tfrac{1}{2}y^2 \quad \text{as} \quad y \to \pm\infty; \end{aligned}\right\} \qquad [2b]$$

the constant f represents the net flux through the slit, with no loss of generality in setting ψ_2 as antisymmetric in x on the plane. If follows that ψ_1 represents the behaviour where the shear flow is in the same direction on both sides of the plane, and ψ_2 has the shear flow in opposite directions on the two sides of the plane. The no-slip conditions are satisfied on the plate $y = 0, |x| > a$. However, it should be emphasized that the conditions for $y = 0, |x| < a$ are only those which are necessary for the symmetries given in the definitions for ψ_1 and ψ_2. In fact, no physical conditions can be set, *a priori*, across the slit except that the velocities and stresses be continuous across $y = 0, |x| < a$, which are the basic physical requirements across any line drawn within the fluid.

Now problems where there is a flow through, or past an aperture are notoriously difficult; for example, the work of Weinbaum (1968) indicated that there is no assurance that a separation streamline will separate from the sharp corner, nor can it be expected that this streamline will be a straightline—the recent study by Higdon (1985), where the numerical solution for the shear flow past a number of different geometries has provided ample, extremely clear evidence of these possibilities. Also Dagan *et al.* (1982) have indicated the problems inherent in fixing the form of the outer boundary conditions when there is a flux through the slit. However, these difficulties evaporate in the present situation where it is seen that we can simply write down the solution

$$\psi_2(x, y) \equiv \tfrac{1}{2}y^2, \quad \forall\, x, y, \tag{3}$$

to the problem defined by [2b]; there is no flux through the slit and the separation streamline $\psi = 0$ is, in fact, just the x-axis. O'Neill (1977) showed for the circular trough that the separation streamline does lie off the x-axis when d is finite, but that it becomes identical with $y = 0$ in the limit as $d \to \infty$.

Next, to satisfy [2a], it is sufficient to write $\psi_1(x, y) = y\chi(x, y)$, where $\nabla^2\chi = 0$, plus

$$\left.\begin{aligned} &\chi = 0 \quad \text{on} \quad y = 0, \quad |x| > a \\ &\chi_y = 0 \quad \text{on} \quad y = 0, \quad |x| < a \\ &\chi \simeq \pm\tfrac{1}{2}y \quad \text{as} \quad y \to \pm\infty. \end{aligned}\right\} \tag{4}$$

For $y \geqslant 0$, we write

$$\chi = \tfrac{1}{2}y + \tilde{\chi}, \tag{5}$$

taking Fourier transforms in x, defined by

$$\bar{\chi}(\alpha, y) = \int_{-\infty}^{\infty} e^{i\alpha x}\tilde{\chi}(x, y)\,dx,$$

we see that the solution of the transformed harmonic equation is

$$\bar{\chi} = 2\pi A(\alpha)e^{-|\alpha|y}, \quad y \geqslant 0,$$

for some function $A(\alpha)$. Satisfying the conditions on $y = 0$ then leads to the dual integral equations:

$$\int_0^{\infty} A(\alpha)\cos\alpha x\,d\alpha = 0, \quad |x| > a; \qquad \int_0^{\infty} \alpha A(\alpha)\cos\alpha x\,d\alpha = \tfrac{1}{4}, \quad |x| < a.$$

This is a standard problem, and the solution can be found in Sneddon (1966):

$$A(\alpha) = \tfrac{1}{4}a\alpha^{-1}J_1(a\alpha),$$

where J_1 is the Bessel function. Taking the inverse transform (Erdelyi 1954), gives, eventually,

$$\psi = \tfrac{1}{2}y^2 + \frac{|y|\,\mathrm{sgn}\,y}{2\sqrt{2}}\{(y^2 - x^2 + a^2) + [(y^2 + x^2)^2 + 2a^2(y^2 - x^2) + a^4]^{\frac{1}{2}}\}^{\frac{1}{2}}, \quad \forall\, x, y; \tag{6}$$

this agrees with the particular solution presented by O'Neill (1977) and Wakiya (1978) as $d \to \infty$. In particular, we note that $u(x, 0) = (a^2 - x^2)^{\frac{1}{2}}/2$ for $|x| < a$; hence, although the velocity is zero at the sharp corner of the slit, the stresses have a square-root singularity there. Further, close to the corner $x = -a, y = 0$ we have

$$\psi \simeq \frac{\sqrt{a}}{2\sqrt{2}}\rho^{\frac{3}{2}}\left(\cos\frac{\phi}{2} - \cos\frac{3\phi}{2}\right)$$

when $x + a = -\rho\cos\phi, y = \rho\sin\phi$ to give the classical solution of Carrier & Lin (1948). The streamlines are sketched in figure 1; the maximum deflection of the streamline $\psi \simeq y^2$ (for large x) is $d(y) = y(1 + y^2)^{\frac{1}{2}} - y^2$ (at $x = 0$), so that $d(0) = 0$ and $d(\infty) = \frac{1}{2}$.

The expression for ψ can be given most simply in terms of elliptic coordinates ξ, η, defined by $x = a\cosh\xi\sin\eta, y = a\sinh\xi\cos\eta$. The plane (with the slit removed) is given by $\eta = \pm\pi/2$ where, generally, $\eta(=\text{const})$ are hyperbolae with $x = \pm a, y = 0$ as foci; $\xi(=\text{const})$ represent ellipses with the same foci. In terms of these coordinates, $\psi = (a^2e^{\xi}\sinh\xi)/2\cdot\cos^2\eta$ in the upper half-plane.

When polar coordinates r, θ are introduced by $x = r\cos\theta, y = r\sin\theta$, it also follows that

$$\psi = \frac{r\sin\theta}{2\sqrt{2}}\{[(a^2 - r^2\cos 2\theta) + (r^4 - 2a^2r^2\cos 2\theta + a^4)^{\frac{1}{2}}]^{\frac{1}{2}} + \sqrt{2}r\sin\theta\}, \tag{7}$$

in the upper half-plane. Of particular interest is the limiting situation as $a \to 0$, and here [7] shows

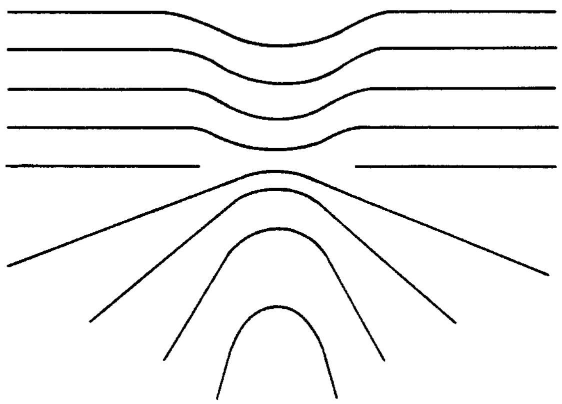

Figure 1. Streamlines for the flow represented by [6]; the shear flow is in the upper half-plane and the flow induced across the slit is in the lower half-plane.

$\psi = [r^2 + (a^2/4)]\sin^2\theta + O(a^4)$. The first term represents the basic shear flow, and the second term is the dominant contribution, of $O(a^2)$, when the slit is narrow. We write

$$\Psi(x, y) = \sin^2\theta = \frac{y^2}{x^2 + y^2} \qquad [8]$$

and observe that this represents a fundamental singularity for the 2-D Stokes equation; the corresponding velocities are

$$U = \Psi_y = -\frac{2x^2 y}{(x^2 + y^2)^2} \quad \text{and} \quad V = -\Psi_x = \frac{2xy^2}{(x^2 + y^2)^2}. \qquad [9]$$

In the upper half-plane the limiting behaviour for small a is given by $\psi = y^2 + (a^2\Psi/4)$ and in the lower half-plane by $-a^2\Psi/4$. The flow represented by Ψ is entirely radial, with $\mathscr{U} = r^{-1}\Psi_\theta = 2r^{-1}\sin 2\theta$ and $\mathscr{V} = -\Psi_r = 0$, where $\mathscr{U}$ and $\mathscr{V}$ are the radial and transverse velocities, respectively. Hence the induced flow in the lower half-plane for small a is radial, with fluid being drawn towards the narrow slit in the third quadrant, and then expelled away from it in the fourth quadrant; all streamlines pass through the origin.

The author is grateful to Professor Allen Chwang for showing the relation of the singularity [8] to the set of singular solutions presented by Chwang & Wu (1975). The function Ψ_S is defined as a unit Stokeslet in the x-direction, and Ψ_R as a unit rotlet in the z-direction (i.e. perpendicular to the x, y-plane) with

$$\Psi_S = y(1 - \ln r) \quad \text{and} \quad \Psi_R = -\ln r. \qquad [10]$$

He then considered the combination

$$\Psi_C = -\frac{\partial \Psi_S}{\partial y} + \Psi_R + 1, \qquad [11]$$

which represents the sum of a Stokes doublet and rotlet, where the constant 1 is added to ensure that $\Psi_C = 0$ on $y = 0$; substitution of [10] into [11] shows that $\Psi_C = \Psi$, as given in [8]. Further, the corresponding velocities $U_S = -\ln r + x^2/r^2$, $V_S = xy/r^2$, $U_R = -y/r^2$ and $V_R = x/r^2$ combine through [10] to give [9], thereby providing agreement with the expressions of Chwang & Wu (1975).

We note, finally, that the singular solution $\psi = \Psi$ is complementary to $\psi \propto \theta - \sin\theta\cos\theta$, which represents the Stokes flow through a narrow slit at the origin in the plane $y = 0$; again, the flow is completely radial with $\mathscr{U} \propto r^{-1}\sin^2\theta$.

One immediate extension is to consider the geometry with an infinite set of equally spaced (narrow) slits in the wall $y = 0$ at $x = na$, for all integers n. This situation would lead to a perturbed flow proportional to

$$\Psi_s = \sum_{n=-\infty}^{\infty} \frac{y^2}{(x - na)^2 + y^2}, \quad y \geqslant 0.$$

Now this series can be summed (cf. Hansen 1975) for

$$\Psi_s = \frac{\pi y}{a}\sinh\left(\frac{2\pi y}{a}\right)\left[\cosh\left(\frac{2\pi y}{a}\right) - \cos\left(\frac{2\pi x}{a}\right)\right]^{-1}, \quad y \geqslant 0,$$

and so $\Psi_s \simeq \pi y/a$ as $y \to \infty$, indicating that a uniform stream has been induced. The streamlines with $\Psi_s \leqslant 1$ all pass through the singular points $x = na, y = 0$, whereas those with $\Psi_s > 1$ are smooth curves with periodic behaviour. The flow in the lower half-plane is sketched in figure 2.

(B) We now investigate the following situation: fluid flows around the circular cylinder $r = 1$, which has a narrow slit at the point $r = 1, \theta = 0$; see figure 3. If there were no slit, then $\psi = 0, r < 1$ and $\psi = r^2 - 2\ln r - 1, r > 1$—we wish to calculate the perturbed flow inside the cylinder when there is a slit.

One approach would be to consider the rotating flow outside a circular arc which subtends a finite angle at the origin, and then take the limiting case as the angle tends to zero. The analysis of Hasimoto (1979), who considered the oblique flow past a circular arc could be adapted for such a situation, though, for a very narrow gap, his result is directly applicable. However, it is found that the limiting case is a rather complex singular perturbation problem where the domain inside the circular arc maps, through the Joukowski transformation, onto a domain which itself has asymptotically small dimensions. Consequently, we can more easily use the basic singular solution Ψ calculated above to find the flow induced inside the cylinder through a narrow slit.

Therefore, we write

$$\psi = \epsilon \sin^2\phi + \psi_p(r, \theta), \quad r \leqslant 1, \tag{12}$$

for the stream function inside the cylinder, where the first term is proportional to Ψ in the new coordinates, as represented in figure 3; ϵ is a constant which is proportional to the square of the width of the slit in the cylinder. Now

$$\sin^2\phi = \frac{1 - 2r\cos\theta + r^2\cos^2\theta}{1 - 2r\cos\theta + r^2}, \quad r \leqslant 1,$$

so that satisfying the necessary boundary conditions $\psi = \psi_r = 0$ on $r = 1$ implies $\psi_p = -\epsilon(1 - \cos\theta)/2$ and $\psi_{pr} = \epsilon(1 + \cos\theta)/2$ on $r = 1$. These conditions lead to a very simple closed form solution which can be found for the biharmonic function $\psi_p(r, \theta)$ in the form $a(r) + b(r)\cos\theta$, and completing the details shows that

$$\psi = \epsilon\left(\frac{1 - 2r\cos\theta + r^2\cos^2\theta}{1 - 2r\cos\theta + r^2} + \frac{1}{4}r^2 - \frac{3}{4} + \frac{1}{2}r\cos\theta\right). \tag{13}$$

In particular, $(\psi_{rr})_{r=1} = \epsilon(1 - \cos\theta)^{-1}$, which is positive for all θ, and so there is no separation from the walls of the cylinder. When the streamlines are plotted they show a set of closed loops within the circle $r \leqslant 1$, all of them passing through the singular point $r = 1, \theta = 0$.

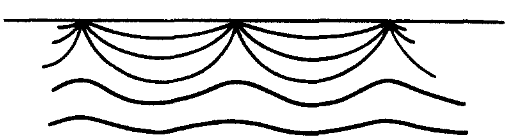

Figure 2. Streamlines for the flow induced in the lower half-plane for a shear flow in the upper half-plane past an infinite set of narrow slits.

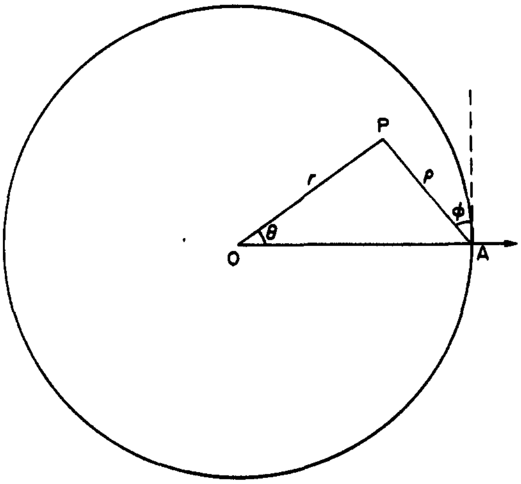

Figure 3. Geometry for the shear flow past a narrow slit in a circular cylinder.

Alternatively, if the basic flow is *inside* the circular cylinder, and a narrow slit at $r = 1, \theta = 0$ leads to a perturbed flow *outside* $r = 1$, then the stream function for $r \geqslant 1$ analogous to [13] is found to be given by

$$\psi = \epsilon \left(\frac{1 - 2r\cos\theta + r^2\cos^2\theta}{1 - 2r\cos\theta + r^2} + \frac{1}{2}\ln r - \frac{1}{2} + \frac{1}{2} r\cos\theta \right). \qquad [14]$$

A uniform stream with velocity $\epsilon/2$ is induced at infinity in the direction corresponding to $\theta = \pi/2$—another example of the so-called Jeffery paradox [see, for example, Dorrepaal *et al.* (1984)]. There are stagnation points at $r = 1, \theta = \pm\pi/2$, and the fluid domain outside the circular cylinder is divided into two distinct regions by the streamlines which pass through these points; all the streamlines to the right pass through the singular point $r = 1, \theta = 0$. Finally, if there are equal-sized slits at the two points $r = 1$ and $\theta = 0, \pi$, then the induced flow at infinity is equivalent to a closed circular cylinder rotating with angular velocity of magnitude ϵ. It is easy to develop other different extensions from the basic solution [14].

(C) To complete the discussion of 2-D flows, we return to the geometry of part A and take the basic linear shear flow in they upper half-plane to be in the z-direction, normal to the x, y-plane. The only resultant velocity component is in the z-direction, and can be written as $w(x, y)$, where w satisfies the harmonic equation $\nabla^2 w = 0$. Further, $w = 0$ on $y = 0, |x| > a, w_y = 0$ on $y = 0, |x| < a$, with $w \simeq y$ as $y \to \infty$ and $w \to 0$ as $y \to -\infty$. The calculations required are identical to those performed earlier to solve [4], and show

$$w = \frac{1}{2\sqrt{2}} [\![\{(y^2 - x^2 + a^2) + [(y^2 + x^2)^2 + 2a^2(y^2 - x^2) + a^2]^{\frac{1}{2}}\}^{\frac{1}{2}} + \sqrt{2}|y| \operatorname{sgn} y]\!].$$

When we take the limit as $a \to 0$, it is seen that

$$w \simeq y + \frac{a^2}{4}\frac{y}{x^2 + y^2} = \left(r + \frac{a^2}{4r} \right)\sin\theta, \quad y \geqslant 0.$$

Hence the singular solution for a linear shear flow parallel to a narrow slit can be represented by

$$W = \frac{y}{x^2 + y^2} = \frac{\sin\theta}{r}, \qquad [15]$$

this represents a potential doublet in the y-direction, with $W = \partial(\ln r)/\partial y$.

It is now straightforward to combine [8] and [15] to compute the effect when there is a general angle between the straight-line narrow slit of infinite length and the direction of the shear flow.

3. FLOWS PAST A CIRCULAR HOLE

(A) To avoid unfamiliar notation, we proceed to redefine all the physical quantities, and there is no overlap between that used in this and the previous section. The basic problem investigated concerns the linear shear flow along a plane which has a circular hole. We introduce a cylindrical coordinate system (ρ, θ, z), where the plane is represented by $z = 0$, and the hole is $z = 0, \rho \leqslant c$. The radial, aximuthal and axial velocities are written as u, v and w respectively, and the shear velocity in the upper half-plane $z \geqslant 0$ is

$$u \simeq 2z\cos\theta, \quad v \simeq -2z\sin\theta, \quad w \simeq 0, \quad z \to \infty; \qquad [16]$$

in the lower half-plane all velocities tend to zero at infinity.

We again divide the solution into symmetric and antisymmetric parts in z, with the latter being given by

$$u_2 = z\cos\theta, \quad v_2 = -z\sin\theta, \quad w_2 = 0, \quad \forall z. \qquad [17]$$

To evaluate the symmetric part, which comprises most of the remainder of this section, it is necessary to solve the Stokes flow equations in the domain $z \geqslant 0$ subject to the conditions $u_1 \simeq z\cos\theta, v_1 \simeq -z\sin\theta, w_1 \simeq 0$ as $z \to \infty$, plus $u_1 = v_1 = w_1 = 0$ on $z = 0, \rho > c$; $u_{1z} = v_{1z} = w_1 = 0$ on $z = 0, \rho < c$. The final solution is then formed by adding $u_1 + u_2, v_1 + v_2$ and $w_1 + w_2$.

We can use the representations introduced by Ranger (1972) in this situation, whereby we write

$$\left.\begin{aligned} u_1 &= [(\rho^{-1}\psi_z)_\rho + \rho^{-2}\chi]\cos\theta, \\ v_1 &= -[\rho^{-2}\psi_z + (\rho^{-1}\chi)_\rho]\sin\theta, \\ w_1 &= -[\rho^{-1}\psi_{\rho\rho} - \rho^{-2}\psi_\rho]\cos\theta, \end{aligned}\right\} \qquad [18]$$

where $\psi(\rho, z)$ and $\chi(\rho, z)$ satisfy $L_{-1}^2(\psi) = 0, L_{-1}(\chi) = 0$, with L_{-1} being the operator

$$L_{-1} \equiv \frac{\partial^2}{\partial\rho^2} - \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial z^2}.$$

Further, it is known from further work by Ranger (1978), where a somewhat similar problem is investigated, that in the solution for ψ it is sufficient to write $\psi = z\phi$, where $\phi(\rho, z)$ just satisfies $L_{-1}(\phi) = 0$. In terms of ϕ and χ we can express the condition at infinity as $\chi \simeq \alpha z\rho^2$, $\phi \simeq \beta z\rho^2$ with α and β constants such that $\alpha + 2\beta = 1$. Consequently, for $z \geqslant 0$, we write

$$\chi = \alpha z\rho^2 + \tilde{\chi} \quad \text{and} \quad \phi = \beta z\rho^2 + \tilde{\phi}, \quad \alpha + 2\beta = 1, \qquad [19]$$

so that $\tilde{\chi}, \tilde{\phi} \to 0$ as $z \to \infty$, and the conditions on $z = 0$ become, most generally,

$$\tilde{\phi} = \tilde{\psi} = \gamma \quad \text{on} \quad z = 0, \quad \rho > c,$$

and

$$\tilde{\phi}_z = (\delta - \beta)\rho^2, \quad \tilde{\chi}_z = -(2\delta + \alpha)\rho^2 \quad \text{on} \quad z = 0, \quad \rho < c,$$

for constants γ and δ.

Both these resultant problems for $\tilde{\phi}$ and $\tilde{\chi}$ are equivalent to solving $L_{-1}(X) = 0$ for $z \geqslant 0$ with $X = \mathrm{A}$ on $z = 0, \rho > c, X_z = -\mathrm{B}\rho^2$ on $z = 0, \rho < c$ for constants A and B. Then X can be written as the Hankel transform

$$X(\rho, z) = \rho \int_0^\infty D(k)J_1(k\rho)\mathrm{e}^{-kz}\mathrm{d}k, \qquad [20]$$

where

$$\left.\begin{aligned} &\int_0^\infty D(k)J_1(k\rho)\mathrm{d}k = \mathrm{A}\rho^{-1}, \quad \rho > c, \\ &\int_0^\infty kD(k)J_1(k\rho)\mathrm{d}k = \mathrm{B}\rho, \quad \rho < c. \end{aligned}\right\} \qquad [21]$$

This pair of integral equations belongs to the case considered by Titchmarsh, and summarized by Sneddon (1966), to show

$$D(k) = \mathrm{A}\frac{\sin kc}{kc} - \frac{4\mathrm{B}c^2}{3\pi k}\left(\sin kc + \frac{3\cos kc}{kc} - \frac{3\sin kc}{k^2c^2}\right). \qquad [22]$$

Taking the inverse transform is not direct, but it can be shown, on adapting different results from Erdelyi (1954), that

$$\int_0^\infty \frac{\sin kc}{kc}J_1(k\rho)\mathrm{e}^{-kz}\,\mathrm{d}k = \frac{1}{\rho} - \frac{\sqrt{2}}{2c\rho}\{[(z^2+\rho^2-c^2)^2 + 4c^2z^2]^{\frac{1}{2}} - (z^2+\rho^2-c^2)\}^{\frac{1}{2}}$$

and

$$\begin{aligned} \int_0^\infty \left(\frac{\sin kc}{kc} + \frac{2\cos kc}{k^2c^2} - \frac{2\sin kc}{k^3c^3}\right)J_1(k\rho)\mathrm{e}^{-kz}\,\mathrm{d}k &= \frac{1}{3\rho} + \frac{\sqrt{2}}{6c^3\rho}(z^2 - 2\rho^2 - c^2) \\ \times \{[(z^2+\rho^2-c^2)^2 + 4c^2z^2]^{\frac{1}{2}} - (z^2+\rho^2-c^2)\}^{\frac{1}{2}} &- \frac{2\sqrt{2}z}{3c^2\rho}\{[(z^2+\rho^2-c^2)^2+4c^2z^2]^{\frac{1}{2}} \\ + (z^2+\rho^2-c^2)\}^{\frac{1}{2}} &+ \frac{\rho z}{c^3}\arcsin\left\{\frac{2c}{[z^2+(\rho-c)^2]^{\frac{1}{2}} + [z^2+(\rho+c)^2]^{\frac{1}{2}}}\right\}. \end{aligned}$$

Substituting these into [20] and [22] gives the complete solution to the boundary value problem for $X(\rho, z)$, and hence for $\tilde{\chi}(\rho, z)$ and $\tilde{\phi}(\rho, z)$ in terms of the constants α, β, γ and δ.

The final step is to evaluate these four constants; one relation is already included in [19]. The remaining relations follow from requiring that the velocities u_1, v_1 and w_1 are bounded in the neighbourhood of the run of the hole where $\rho = c, z = 0$. Completing the detailed, but standard analysis finally yields

$$\alpha = \beta = \tfrac{1}{3}, \quad \gamma = -\tfrac{1}{6}\pi^{-1}c^3, \quad \delta = 0.$$

Consequently,

$$\begin{aligned}\phi(\rho, z) = \chi(\rho, z) = \frac{1}{3}z\rho^2 - \frac{c^3}{6\pi} - \frac{2z\rho^2}{3\pi} \arcsin\left\{\frac{2c}{[z^2+(\rho-c)^2]^{\frac{1}{2}} + [z^2+(\rho+c)^2]^{\frac{1}{2}}}\right\} \\ + \frac{4\sqrt{2}cz}{9\pi}\{[(z^2+\rho^2-c^2)^2+4c^2z^2]^{\frac{1}{2}} + (z^2+\rho^2-c^2)\}^{\frac{1}{2}} \\ - \frac{\sqrt{2}}{9\pi}(z^2 - 2\rho^2 - \frac{3}{4}c^2)\{[(z^2+\rho^2-c^2)^2+4c^2z^2]^{\frac{1}{2}} - (z^2+\rho^2-c^2)\}^{\frac{1}{2}},\end{aligned} \quad [23]$$

from which the velocities can be found through [18] to complete the solution for the symmetric part of the flow. This is a new solution, in a reasonably simple, closed form, of the Stokes equations.

Of particular interest for subsequent work is the limiting form of [23] for small c, which is given by

$$\phi = \chi \simeq \frac{1}{3}z\rho^2 - \frac{c^3}{6\pi}\left[1 - \frac{z}{(\rho^2+z^2)^{\frac{1}{2}}}\right], \quad z \geqslant 0. \quad [24]$$

When the basic linear shear flow is subtracted, it then follows that the dominant contribution to the perturbed velocities for the flow in the upper half-plane is given by

$$u \simeq -\frac{c^3 z\rho^2 \cos\theta}{2\pi(\rho^2+z^2)^{\frac{5}{2}}}, \quad v \simeq 0, \quad w \simeq -\frac{c^3 z^2 \rho \cos\theta}{2\pi(\rho^2+z^2)^{\frac{5}{2}}}; \quad [25]$$

in the lower half-plane the dominant contributions are u, v and $-w$.

When a spherical coordinate system (R, ω, θ) is introduced by $\rho = R\sin\omega$, $z = R\cos\omega$, and the velocities in the R, ω and θ directions are written as $\mathscr{U}$, $\mathscr{V}$ and $\mathscr{W}$ respectively, then [25] can be expressed by

$$\mathscr{U} = -\frac{c^3}{4\pi}\frac{\sin 2\omega \cos\theta}{R^2} \quad \text{and} \quad \mathscr{V} = \mathscr{W} = 0; \quad [26]$$

the flow is completely radial in the spherical sense. The result [26] represents the fundamental singular solution for the shear flow past a small hole; the velocity introduced is proportional to the cube of the radius of the hole.

In the lower half-plane the fluid upstream of the hole is drawn in towards the hole, and then forced away on the downstream side. The mass flux in the direction of $\theta = 0$ is proportional to

$$\int_0^\infty dz \int_0^\pi \rho u \sin\theta \, d\theta,$$

which is a constant from [25]. If we use the representation (cf. Ranger 1973)

$$\mathscr{U} = -\frac{\cos\theta}{R^2}\left(\frac{1}{\sin\omega}\Phi_\omega\right)_\omega, \quad \mathscr{V} = \frac{\cos\theta}{R}\left(\frac{1}{\sin\theta}\Phi_R\right)_\omega \quad \text{and} \quad \mathscr{W} = -\frac{\sin\theta}{R}\frac{\Phi_R}{\sin^2\omega},$$

for solutions of the Stokes equations in spherical coordinates, then [26] corresponds to

$$\Phi \propto -\cos\omega \sin^2\omega, \quad [27]$$

which is clearly a solution of $L^2_{-1}(\Phi) = 0$. The corresponding axisymmetric Stokes flow represents the source flow through a hole as given, for example, in Happel & Brenner (1983).

Professor Chwang kindly also calculated the representation for this 3-D singular solution [25] in terms of those catalogued by Chwang & Wu (1975), and the author again wishes to acknowledge his contribution. If $\mathbf{u}_S$ represents the velocity vector for a unit (3-D) Stokeslet in the x-direction, and $\mathbf{u}_R$ for a unit (3-D) rotlet in the y-direction, then

$$\mathbf{u}_S = \frac{\hat{e}_x}{R} + \frac{x\mathbf{R}}{R^3} \quad \text{and} \quad \mathbf{u}_R = \frac{\hat{e}_y\mathbf{R}}{R^3}. \qquad [28]$$

Then he showed that the combination of a Stokes doublet and rotlet, defined by

$$\mathbf{u}_C = \frac{\partial \mathbf{u}_S}{\partial z} + \mathbf{u}_R,$$

leads through [28] to

$$\mathbf{u}_C = -\frac{3xz\mathbf{R}}{R^5},$$

which, in component form, shows

$$u_C = -\frac{3z\rho^2 \cos\theta}{R^5}, \quad v_C = 0, \quad w_C = -\frac{3\rho z^2 \cos\theta}{R^5};$$

this is exactly the same as [25] when the strengths of the doublet and rotlet are $c^3/6\pi$.

(B) We can develop the analogue in 3-D for the situation considered in section 2B. A sphere of unit radius has a small hole on the rim at the point $R = 1, \theta = \pi$ in a spherical coordinate system with origin at the centre of the sphere. A flow past the sphere will then lead to a weak induced flow in the interior, which can be modelled by the singular solution [26] based at the hole. It is possible to proceed as was done in section 2B; however, here we take the alternative procedure of evaluating an asymptotic approximation of a previously calculated solution. Dorrepaal (1976) considered the asymmetric flow past a spherical cap, and when the angle of the cap tends to π, the flow we require will be described by the limiting form of his solution.

We work directly with his notation for the remainder of this section. On p. 741 we put $\beta = \pi - \alpha$ where $\beta \ll 1$. Then the coefficients show

$$A_n^{(1)} = O(\beta^5),$$
$$A_n^{(2)} = \frac{(-1)^n}{12\pi}\beta^3 + O(\beta^5),$$
$$A_n^{(3)} = \frac{(-1)^n}{6\pi}\beta^3 + O(\beta^5) \quad \text{for} \quad n = 1, 2, \ldots,$$

with the exception that $\frac{1}{3}$ is added to $A_1^{(1)}$ and $A_1^{(2)}$; also, $h = -1 + O(\beta^5)$. Consequently, the resulting series can be summed to give the functions

$$\psi(r,\theta) = \frac{1}{12\pi}\beta^3(1-r^2)\sigma(r,\theta) \quad \text{and} \quad V_3(r,\theta) = \frac{1}{6\pi}\beta^3\sigma(r,\theta),$$

where

$$\sigma(r,\theta) = \frac{1-r^2}{(1+2r\cos\theta+r^2)^{\frac{1}{2}}} - (1 - r\cos\theta), \qquad [29]$$

(in Dorrepaal's notation still) to give a simple closed-form solution to the problem. The first term in [29] is equivalent to the singular solution [26] in the neighbourhood of the hole. In particular, the velocity on the centreline $\theta = 0, \pi$ (called $q_{(\theta)}$ by Dorrepaal) can be found as

$$q_{(\theta)} = -\frac{1}{8\pi}\beta^3(1-r)(2+r)(1+r)^{-1}\cos\phi.$$

There is clearly no free eddy in this 3-D situation, and all the fluid particles traverse closed loops inside the sphere, passing through the singular point at the hole.

4. FLOWS PAST FINITE SLITS

We next consider the linear shear flow along an infinite plate in which there is a narrow slit of finite length. It is seen how to use the results gained earlier to motivate particular assumptions which guide steps taken in the solution. In each case a formal proof would require an involved, and highly detailed analytical calculation; these are not considered here, to aid physical understanding and for brevity. Specifically, we calculate the singular solution which represents the flow in the lower half-plane when the external shear flow is present in the upper half-plane, taking the slit to be a straight line first perpendicular, and then parallel to the direction of the shear flow.

(A) In the first situation, the narrow slit is positioned in the plane $z = 0$ along the y-axis for $|y| < 1$; the given shear flow in the upper half-plane is in the x-direction with velocity proportional to z. This situation is therefore intermediary between the limiting case discussed in section 2A when the slit has infinite length and that of section 3A when it has zero length. Consequently, we can assume that the streamlines for the flow lie completely in planes of the form $z/x = \text{const}$, all of which include the y-axis.

We now construct a cylindrical coordinate system $O(X, Y, \alpha)$, where $x = X \sin \alpha$, $y = Y$ and $z = X \cos \alpha$, let $U \sin 2\alpha$ and $V \sin 2\alpha$ be the velocities in the X-(radial) and Y-(axial) directions; the angular velocity in the α-direction is zero. These representations for the velocities automatically give no-slip conditions on the plate $\alpha = 0, \pi$. The equation of continuity becomes

$$U_X + X^{-1} U + V_Y = 0$$

with these representations, and so a stream function $\psi(X, Y)$ exists such that $U = X^{-1}\psi_Y$, $V = -X^{-1}\psi_X$. Further, the momentum equations in the X- and Y-directions are

$$U_{XX} - X^{-1} U_X - 3X^{-2} U + U_{YY} = 0,$$

and

$$V_{xx} + X^{-1} V_X - 4X^{-2} V + V_{YY} - 2X^{-1} U_Y = 0,$$

which show consistency when the stream function ψ satisfies

$$\psi_{XX} - 3X^{-1}\psi_X + \psi_{YY} = 0. \qquad [30]$$

The final momentum equation shows that the pressure is given by $p = 2X^{-1} U \sin 2\alpha$. The earlier solutions [8] and [25] can now be seen as special cases; $\psi \propto Y$ corresponds to the effect of an infinite slit, and $\psi \propto 3Y(X^2 + Y^2)^{-\frac{1}{2}} - Y^3(X^2 + Y^2)^{-\frac{3}{2}}$ to the effect of a hole. (The latter expression becomes $\psi \propto 3 \sin \gamma - \sin^3 \gamma$, where $Y/X = \tan \gamma$.)

To understand the effect of the finite slit we must find solutions [30], and to this end we consider the existence of solutions of the form $\psi = F(\eta)$ where $X = \sinh \xi \cos \eta$, $Y = \cosh \xi \sin \eta$; this would indicate that the streamlines follow confocal hyperbolae in the X, Y-plane, with foci at $X = 0$, $Y = \pm 1$. The support for such a conjecture is well-established in known solutions for both the harmonic and biharmonic equations. We find, on substitution, that such solutions do in fact exist with $F(\eta)$ given by $F'' + 3 \tan \eta F' = 0$; hence

$$F(\eta) \propto 3 \sin \eta - \sin^3 \eta, \qquad [31]$$

to agree with [26] as $X, Y \to \infty$ (i.e. $\xi \to \infty$). In the original coordinate system we have

$$2 \sin^2 \eta = (x^2 + Y^2 + 1) - [(X^2 + Y^2 - 1)^2 + 4X^2]^{\frac{1}{2}}.$$

The above discussion shows the existence of the solution [31] which satisfies the Stokes equations, the no-slip conditions on the plate and has appropriate singular behaviour on the narrow slit. (To formally prove this is the only solution would require a process such as considering the flow past a hole with an elliptical profile, thereby generalizing section 3A, and then taking the limit as the area of the ellipse tends to zero, maintaining the same foci.)

The fluid in the lower half-plane is drawn towards the slit in planes with constant radial angles, and the streamlines followed are confocal hyperbolae within these planes, which therefore meet the slit perpendicularly. As $X \to 0$ along the slit $|Y| < 1$, the normal velocity U becomes infinite as X^{-1}, but the velocity along the slit satisfies $V_X = 0$ for $X = 0, |Y| < 1$; both U and V are zero for $X = 0, |Y| > 1$.

(B) In the second case, the narrow slit is placed along the x-axis for $|x| < 1$; the shear flow is still in the x-direction with velocity proportional to z. Here we are intermediary between the cases described in sections 2C (with a change of notation) and 3A. In this situation we consider the possibility that the streamlines lie in planes of the form $z/y = \text{const}$. We can now repeat the equivalent calculations to those done above—here in an abbreviated form.

The cylindrical coordinate system $O(X, Y, \beta)$ (quite separate from that of part A) has $x = X, y = Y \sin\beta, z = Y\cos\beta$ and the velocities in the Y-(radial) direction and X-(axial) directions are $U\cos\beta$ and $V\cos\beta$, respectively. The radial and axial momentum equations show

$$U_{YY} - Y^{-1}U_Y + U_{XX} = 0 \quad \text{and} \quad V_{YY} + Y^{-1}V_Y - Y^{-2}V + V_{XX} = 2Y^{-1}U_X$$

with the pressure $p = 2Y^{-1}U\cos\beta$ from the angular momentum equation. These momentum equations are consistent when the stream function defined by $U = Y^{-1}\psi_X$ and $V = -Y^{-1}\psi_Y$ satisfies the equation

$$\psi_{YY} - 3Y^{-1}\psi_Y + 3Y^{-2}\psi + \psi_{XX} = 0. \qquad [32]$$

Special cases of solutions for this equation are $\psi \propto Y$, which is equivalent to [13] (with the change of notation), and $\psi \propto Y^3(X^2 + Y^2)^{-\frac{3}{2}}$, which is equivalent to [25]. To solve [32], we first write $\psi(X, Y) = Y\phi(X, Y)$, which requires $\phi_{YY} - Y^{-1}\phi_Y + \phi_{XX} = 0$, and so leads to the general solution

$$\psi(X, Y) = Y^2 \int_0^\infty E(\alpha)K_1(\alpha Y)\cos\alpha X \,\mathrm{d}\alpha \qquad [33]$$

for some arbitrary function $E(\alpha)$; K_1 represents a modified Bessel function. It is difficult to be certain of the exact form of the boundary conditions on $Y = 0$, but those which follow from taking the limit as the width of the slit tends to zero from the solution obtained for the flow parallel to the infinite slit in section 2, part C show $\psi_Y \propto \text{const}$ on $Y = 0, |X| < 1$ and $\psi_Y = 0$ on $Y = 0, |X| > 1$. With these conditions it follows that

$$E(\alpha) \propto \sin\alpha/\alpha.$$

Evaluating the integral [33] which results from this form for $E(\alpha)$ shows

$$\psi(X, Y) \propto Y\left\{\frac{X+1}{[(X+1)^2 + Y^2]^{\frac{1}{2}}} - \frac{X-1}{[(X-1)^2 + Y^2]^{\frac{1}{2}}}\right\}; \qquad [34]$$

and the behaviour $\psi \propto Y^3(X^2 + Y^2)^{-\frac{3}{2}}$ as $X, Y \to \infty$ is recovered, as necessary. The velocities are bounded everywhere except in the neighbourhood of the tips of the slits where they grow as R^{-1} when $R = [(X \pm 1)^2 + Y^2]^{\frac{1}{2}}$. The streamlines in the azimuthal plane are smooth curves which are radial (in the spherical sense) at infinity.

5. TWO-PHASE FLUID RESULTS

The foregoing results can be extended to the two-phase situation where the fluid in the upper half-space has viscosity μ_1, and that in the lower half-space has viscosity μ_2. We reconsider the problem of section 2A alone, but a general result is gained which can be immediately extended to the other situations.

We write $\psi_1(x, y)$ and $\psi_2(x, y)$ as the stream functions in the half-planes $y > 0$ and $y < 0$ respectively, and require $\psi_1 \simeq y^2$ as $y \to +\infty, \psi_2 \to 0$ as $y \to -\infty$. On the line $y = 0, |x| > a$, we have $\psi_1 = \psi_{1y} = 0$ plus $\psi_2 = \psi_{2y} = 0$ as the no-slip conditions. On the interface $y = 0, |x| < a$, it is necessary that the velocity is continuous for $\psi_1 = \psi_2$ and $\psi_{1y} = \psi_{2y}$, and finally that the components of stress on the surface are also continuous, which require $\mu_1(\psi_{1yy} - \psi_{1xx}) = \mu_2(\psi_{2yy} - \psi_{2xx})$ plus $p_1 + 2\mu_1\psi_{1xy} = p_2 + 2\mu_2\psi_{2xy}$, where p represents the pressure.

It is sufficient to write $\psi_i(x, y) = y\phi_i(x, y)$ for $i = 1, 2$, where $\nabla^2\phi_i = 0$, and expressing the above conditions in terms of ϕ_i leads to

$$\left.\begin{aligned} &\phi_1 = \phi_2 = 0 \quad \text{on} \quad y = 0, \quad |x| > a, \\ &\phi_1 = \phi_2, \quad \mu_1\phi_{1y} = \mu_2\phi_{2y} \quad \text{on} \quad y = 0, \quad |x| < a. \end{aligned}\right\} \qquad [35]$$

To find the potential functions ϕ_1 and ϕ_2, we write

$$\phi_1 = y + \int_0^\infty A_1(\alpha)\mathrm{e}^{-|\alpha|y - i\alpha x}\,\mathrm{d}\alpha, \quad y \geqslant 0,$$

and

$$\phi_2 = \int_0^\infty A_2(\alpha)\mathrm{e}^{|\alpha|y - i\alpha x}\,\mathrm{d}\alpha, \quad y \leqslant 0;$$

on solving the resultant pair of dual integral equations it follows that

$$A_1(\alpha) = A_2(\alpha) = \frac{a\mu_1}{\mu_1 + \mu_2}\frac{J_1(a\alpha)}{2\alpha}. \qquad [36]$$

The final solutions can be obtained on evaluating the equivalent integrals to those in section 1A. Consequently, the streamlines in the lower half-plane are exactly those found previously for the homogeneous fluid; the only difference being that the velocities are now multiplied by the factor $2\mu_1(\mu_1 + \mu_2)^{-1}$.

Hence, the sketch of the streamlines in figure 1 is still correct for $y < 0$ in the two-phase flow, and for $y \geqslant 0$ the only change is that the deflection of the streamlines from the horizontal lines must be multiplied by $2\mu_1(\mu_1 + \mu_2)$ throughout.

This last statement is still valid in the 3-D situation for the flows described in sections 2C, 3A and 4.

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